

Continuum damage mechanics revised A principle for mechanical and thermal equivalence

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Abstract

A framework is derived for developing constitutive laws for engineering materials. The framework is based on physically motivated assumptions on the mechanical and thermal behaviour of plastically deformed and damaged materials. These assumptions are the starting point in a derivation of the relevant thermodynamic quantities. The procedure reveals important cross-dependencies that have to be considered while developing constitutive equations. A major result of this study is that both deterioration (increase of damage) and healing (decrease of damage) can be modelled in an integrated manner using the same constitutive law. For greater clarity the description is limited to small strain and isotropic behaviour.

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1. Introduction

With continuum damage mechanics (CDM), fracture is viewed as a process developing locally within a continuum. The field originated with the seminal paper by Kachanov (1958) dealing with brittle creep rupture. To model this process, Kachanov introduces a field variable ψ denoted continuity or soundness. With the initial value $\psi = 1$ for a virgin material ψ decreases according to a constitutive law. When $\psi = 0$ fracture is stated to occur. Odqvist and Hult (1962) show that Kachanov's concept implies the linear life-fraction rule of Robinson (1952). Moreover, with A as the load bearing area in uniaxial tension, Odqvist and Hult (1962) interpret ψA as the remaining load bearing area. They also denote $1 - \psi$ "damage". Rabotnov (1963) extends Kachanov's idea by incorporating the effect of damage on the strain rate in creep. Rabotnov also changes variable to $\omega \equiv 1 - \psi$ and gives a very influential interpretation of ω as the area fraction of cavities in a cross section of a uniaxially stressed specimen. Jansson and Stigh (1985) show that this interpretation is misleading if taken literally. Furthermore, they show that the phenomenological

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concept of Kachanov–Rabotnov is consistent with results from micromechanical modelling. Lemaitre (1972) extends the phenomenological models of creep fracture to models for fatigue. In a paper dealing with the development of damage ahead of a crack tip, Janson and Hult (1977) named the field “Continuum Damage Mechanics”.

In the 1980s Chaboche, Lemaitre and co-workers develop methods to derive constitutive laws based on a thermodynamic framework and a principle of strain equivalence, cf. e.g. Lemaitre (1985) and Lemaitre and Chaboche (1990). Although the precise phrasing of the principle varies somewhat, the basic idea is that the damaged material behaves as the virgin material if this material is acted upon with an effective stress, $\tilde{\sigma}$; this stress being an amplified version of the Cauchy stress, σ . With a scalar damage variable, ω , $\tilde{\sigma}$ is defined through

$$\sigma = (1 - \omega)\tilde{\sigma}. \quad (1)$$

The starting point in these theories is the identification of the state variables and a statement of the form of the Helmholtz free energy per unit volume; in the sequel denoted *the free energy*. With f^e and f^p denoting the elastic and plastic parts of the free energy respectively, two different forms are today, almost exclusively, in use

$$f(\epsilon^e, \mathbf{p}, \omega) = (1 - \omega)f^e(\epsilon^e) + f^p(\mathbf{p}), \quad (2)$$

used by e.g. Lemaitre and Chaboche (1990) and

$$f(\epsilon^e, \mathbf{p}, \omega) = (1 - \omega)[f^e(\epsilon^e) + f^p(\mathbf{p})], \quad (3)$$

used by e.g. Ju (1989). Here, ϵ^e and \mathbf{p} denote the elastic strain tensor and the internal variables used to model the *plastic* deformation of the material, respectively. A simple choice for \mathbf{p} is to use the accumulated von Mises-effective plastic strain though we will, for generality, keep the symbol \mathbf{p} .

Both forms of the free energy yield the same structure for the Cauchy stress, i.e. Eq. (1). However, the “stresses” conjugated to plasticity and damage differ; i.e. the plastic stress, defined by $\mathbf{P} \equiv \rho \partial f / \partial \mathbf{p}$, and the damage stress, defined by $\Omega = \rho \partial f / \partial \omega$, differ. With Eq. (2), the plastic stress is given by $\rho \partial f^p / \partial \mathbf{p}$ and with Eq. (3) it is given by $\mathbf{P} = \rho(1 - \omega) \partial f^p / \partial \mathbf{p} = (1 - \omega)\tilde{\mathbf{P}}$. Thus, the last form yields the plastic stress in an effective form, $\tilde{\mathbf{P}}$. The damage stress takes the form $-\rho f^e$ with Eq. (2) and $-\rho(f^e + f^p)$ with Eq. (3). Thus, with Eq. (2) only the energy associated with elastic strain “drives” damage while both forms of the energy contributes if Eq. (3) is used, cf. Olsson and Ristinmaa (2003).

In this paper we will derive a framework for developing constitutive laws starting with physically motivated assumptions on the mechanical and thermal properties of plastically deformed and damaged materials. From these assumptions we will *derive* the form of the free energy. In order to gain clarity, only small deformations and isotropy are considered. Thus, the density, ρ , at a material point is considered to be a constant.

The paper is organized in the following way; first an introduction to relevant parts of the thermodynamics of irreversible processes is given, then results for thermoelastic materials are summarized. The main part of the paper is devoted to extensions of this material model to plasticity and damage. Two simple examples are provided. These examples show some of the capabilities of the present framework. Some further interpretations are offered in the concluding section.

2. Thermodynamic background

In this section a brief introduction is given to thermodynamics of irreversible processes relevant for the present purpose. There is a vast amount of literature available and we do not attempt to give a review of the field. For a general treatise we refer to Kondepudi and Prigogine (1998) and for more specialized works to

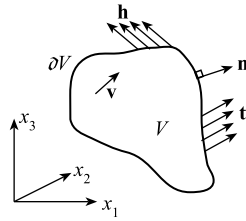


Fig. 1. Body V with surface ∂V and outer normal \mathbf{n} . On the surface traction vector \mathbf{t} and heat flux vector \mathbf{h} act. The material points have velocity vector \mathbf{v} .

Maugin (1992) and Nguyen (2000). The present account differs somewhat from these in order to provide a possibility to obtain an intuitive interpretation to the concept of dissipation.

A body, V , is considered, cf. Fig. 1. On the surface, ∂V , with outer unit normal vector, \mathbf{n} , a traction vector \mathbf{t} acts. During a time interval, Δt , the body is taken through a process which is slow enough to guaranty that the state is always locally well-defined. This is known as the assumption of *local equilibrium*, cf. e.g. Kondepudi and Prigogine (1998). During the process, heat is allowed to pass through the surface; the heat flux vector is denoted \mathbf{h} . No other sources of change of energy are considered. The change in (internal) energy, U , is then given by the sum of the work done by the traction and the heat received from the environment,

$$\Delta U = \int_{\Delta t} \left[\int_{\partial V} t_i v_i d(\partial V) - \int_{\partial V} h_i n_i d(\partial V) \right] dt, \quad (4)$$

where, \mathbf{v} is the velocity vector. Noting that the time interval is arbitrary we can identify the expression in the block parenthesis as the rate of change of energy. Using Cauchy's formula, $t_i = \sigma_{ji} n_j$, the divergence theorem and the principle of local equilibrium we arrive at,

$$\int_V \rho \dot{u} dV = \int_V (\sigma_{ji} v_{i,j}) dV - \int_V h_{i,i} dV, \quad (5)$$

where u is the (internal) energy per unit mass. With the symmetric strain rate tensor $\dot{\epsilon}_{ij} \equiv (1/2)(v_{i,j} + v_{j,i})$, and the equilibrium equations $\sigma_{ji,j} = 0$ and $\sigma_{ij} = \sigma_{ji}$, we can now identify the rate of work per unit volume $\rho \dot{w} \equiv \sigma_{ij} \dot{\epsilon}_{ij}$ and the heat supply per unit volume $\rho \dot{q} \equiv -h_{i,i}$. Noting that the choice of volume element is arbitrary by the principle of local equilibrium we arrive at,

$$\dot{u} = \dot{w} + \dot{q}. \quad (6)$$

The first law of thermodynamics states that the energy is determined solely by the state of the material. With the restrictions imposed here, the rate of change of u is given by Eq. (6).

For the purpose of developing isotropic constitutive laws, the rate of work is conveniently split into two parts by splitting the stress and strain rate tensors in volumetric and deviatoric parts, respectively

$$\rho \dot{w} = \tau_{ij} \dot{\epsilon}_{ij} + \sigma_m \dot{\epsilon}_v. \quad (7)$$

where, $\tau_{ij} \equiv \sigma_{ij} - \delta_{ij} \sigma_m$ and $\dot{\epsilon}_{ij} \equiv \dot{\epsilon}_{ij} - \delta_{ij} \dot{\epsilon}_v/3$ are the deviatoric stress and strain rate tensors, respectively. Furthermore, $\sigma_m = \sigma_{kk}/3$ and $\dot{\epsilon}_v = \dot{\epsilon}_{kk}$ are the mean stress and the volumetric strain rate, respectively.

The second law of thermodynamics states that the internal production of entropy is never negative. Thus, in the process

$$\Delta S + \int_{\Delta t} \left[\int_{\partial V} \frac{h_i n_i}{T} d(\partial V) \right] dt \geq 0. \quad (8)$$

Here, ΔS denotes the total change of entropy of the body and T the absolute temperature. Using the principle of local equilibrium, the rate of change of entropy can be written as the integral of the rate of entropy production per unit volume, $\rho\dot{s}$, over the volume,

$$\int_{\Delta t} \left[\int_V \rho\dot{s} dV + \int_V \left(\frac{h_i}{T} \right)_i dV \right] dt \geq 0, \quad (9)$$

where the divergence theorem has been used on the volume integral of Eq. (8). Noting that both the time interval and the part of the body considered are arbitrary, we arrive at the local form of the second law, $\rho\dot{s} + (h_i/T)_i \geq 0$. Expanding the partial derivative yields

$$\rho\dot{s} - \frac{\rho\dot{q}}{T} - \frac{h_i T_{,i}}{T^2} \geq 0, \quad (10)$$

which is known as the Clausius–Duhem inequality.

For an *imagined* reversible process leading to the same state as the real one, a special rate of heat, $\rho\dot{q}_0$, is identified from Eq. (10). For an “infinitely” slow reversible process ($\mathbf{h} = \mathbf{0}$) the inequality of Eq. (10) is changed to an equality. Thus,

$$\dot{q}_0 = T\dot{s}. \quad (11)$$

With this procedure a reversible rate of heat is associated with any process without reverting to classical thermodynamics, i.e. claiming that every irreversible transformation that occurs in nature can also be achieved through a reversible process.

Since the imagined reversible process and the actual process lead to the same state, a corresponding *reversible* rate of work, \dot{w}_0 , is identified through the first law, Eq. (6)

$$(\dot{u} =) \dot{w}_0 + \dot{q}_0 = \dot{w} + \dot{q}. \quad (12)$$

Equations (11) and (12) are visualized in Fig. 2.

We now introduce the dissipation per unit mass, \dot{d} , through the difference between the rates of work for the actual process and for the associated reversible process,

$$\dot{d} \equiv \dot{w} - \dot{w}_0 = \dot{q}_0 - \dot{q}, \quad (13)$$

where the last equality follows from Eq. (12). With Eqs. (11)–(13), Eq. (10) is rewritten as the *dissipation inequality*,

$$\rho\dot{d} - \frac{h_i}{T} T_{,i} \geq 0. \quad (14)$$

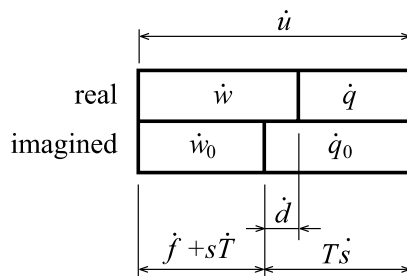


Fig. 2. Visualization of relations between different rates associated with the real and the reversible processes.

Almost exclusively, Fourier's law of heat conduction is used in constitutive modelling of engineering materials, i.e. $h_i = -kT_{,i}$ with $k > 0$. This guarantees that the second term of Eq. (14) gives a non-negative contribution to the dissipation inequality. It thus remains to develop constitutive laws yielding

$$\dot{d} \geq 0. \quad (15)$$

Thus, Clausius–Duhem's inequality reduces to the statement that the rate of work per unit volume is never less than the associated reversible rate of work, cf. Eq. (13). The relations between the reversible and actual processes are visualized in Fig. 2.

When considering strain-based descriptions of non-isothermal processes, it is convenient to use the Helmholtz free energy per unit mass, f , instead of u . The free energy is defined by

$$f \equiv u - sT, \quad (16)$$

and is thus a state function. Taking the time derivative and using Eqs. (6), (11) and (13) we arrive at

$$\dot{f} = \dot{w}_0 - s\dot{T}. \quad (17)$$

Thus, the free energy can be interpreted as the energy available to do work in an isothermal reversible process. Due to dissipation, the rate of work is never less than the rate of free energy in an isothermal process, cf. Fig. 2.

In Section 3, we will use the property that f and s are state functions and introduce principles of thermal and mechanical equivalence to derive the possible forms of f and s consistent with these principles.

3. Constitutive models

Macroscopically, plasticity and damage affect the response of a body in different ways. As an example, consider a load cycle where the external load is first increased from zero and then decreased to zero, cf. Fig. 3. In this load cycle, plasticity is manifested through a permanent deformation while damage shows up as a decrease of the elastic moduli upon unloading. On a micromechanical scale, many different mechanisms are responsible for plasticity and damage. For example, crystal slip by moving dislocations yields a permanent set and is modelled as plasticity; nucleation and growth of microcracks and voids lead to a decrease of the moduli and are modelled as damage.

In this section, internal state variables reflecting inelastic deformation are introduced. This is done in two steps: first a standard model for isotropic thermoelastic materials is modified to allow for *plastic* deformation by introducing state variables, \mathbf{p} , reflecting the state of internal rearrangement due to plastic deformation. In the second step, this model is modified to allow for *damage* by introducing state variables, ω , reflecting the state of internal deterioration. It should be noted that both \mathbf{p} and ω can be considered as *collections* of internal, generally tensorial, state variables. Thus, the representation is general and includes

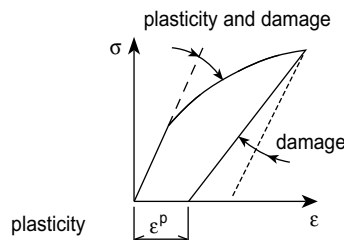


Fig. 3. Load cycle indicating the different manifestations of plasticity and damage.

the possibility to use only one scalar for each of the mechanisms. We will refrain from studying specific forms of the evolution equations for \mathbf{p} and $\boldsymbol{\omega}$ and only indicate possible forms consistent with the dissipation inequality, Eq. (15). The specific forms of these equations determine if the final constitutive equations model time-independent or time-dependent deformation.

Using the following set of state variables, \mathbf{e}^e , ϵ_v^e , T , \mathbf{p} and $\boldsymbol{\omega}$, the rate of the free energy is given by

$$\rho \dot{f} = \rho \frac{\partial f}{\partial \mathbf{e}^e} \cdot \dot{\mathbf{e}}^e + \rho \frac{\partial f}{\partial \epsilon_v^e} \dot{\epsilon}_v^e + \rho \frac{\partial f}{\partial T} \dot{T} + \rho \frac{\partial f}{\partial \mathbf{p}} \cdot \dot{\mathbf{p}} + \rho \frac{\partial f}{\partial \boldsymbol{\omega}} \cdot \dot{\boldsymbol{\omega}}. \quad (18)$$

The rate of reversible work and the entropy are identified by comparison with Eq. (17)

$$\rho \dot{w}_0 = \rho \frac{\partial f}{\partial \mathbf{e}^e} \cdot \dot{\mathbf{e}}^e + \rho \frac{\partial f}{\partial \epsilon_v^e} \dot{\epsilon}_v^e + \rho \frac{\partial f}{\partial \mathbf{p}} \cdot \dot{\mathbf{p}} + \rho \frac{\partial f}{\partial \boldsymbol{\omega}} \cdot \dot{\boldsymbol{\omega}}, \quad (19)$$

and

$$s = -\frac{\partial f}{\partial T}. \quad (20)$$

The reversible rate of work is now conveniently defined by

$$\rho \dot{w}_0 \equiv \boldsymbol{\tau} \cdot \dot{\mathbf{e}}^e + \sigma_m \dot{\epsilon}_v^e + \mathbf{P} \cdot \dot{\mathbf{p}} + \boldsymbol{\Omega} \cdot \dot{\boldsymbol{\omega}}, \quad (21)$$

where \mathbf{P} and $\boldsymbol{\Omega}$ are the “plastic stress” and “damage stress” conjugated to \mathbf{p} and $\boldsymbol{\omega}$, respectively. Identification with Eq. (19) yields

$$\boldsymbol{\tau} = \rho \frac{\partial f}{\partial \mathbf{e}^e}, \quad \sigma_m = \rho \frac{\partial f}{\partial \epsilon_v^e}, \quad \mathbf{P} = \rho \frac{\partial f}{\partial \mathbf{p}}, \quad \boldsymbol{\Omega} = \rho \frac{\partial f}{\partial \boldsymbol{\omega}}. \quad (22a,b,c,d)$$

The associated rate of dissipation is derived from Eqs. (7), (13) and (21)

$$\rho \dot{d} = \boldsymbol{\tau} \cdot (\dot{\mathbf{e}} - \dot{\mathbf{e}}^e) + \sigma_m (\dot{\epsilon}_v - \dot{\epsilon}_v^e) - \mathbf{P} \cdot \dot{\mathbf{p}} - \boldsymbol{\Omega} \cdot \dot{\boldsymbol{\omega}}. \quad (23)$$

Equation (20) yields the rate of entropy

$$\rho \dot{s} = -\rho \frac{\partial^2 f}{\partial T \partial \mathbf{e}^e} \cdot \dot{\mathbf{e}}^e - \rho \frac{\partial^2 f}{\partial T \partial \epsilon_v^e} \dot{\epsilon}_v^e + \rho \frac{\partial s}{\partial T} \dot{T} - \rho \frac{\partial^2 f}{\partial T \partial \mathbf{p}} \cdot \dot{\mathbf{p}} - \rho \frac{\partial^2 f}{\partial T \partial \boldsymbol{\omega}} \cdot \dot{\boldsymbol{\omega}}, \quad (24)$$

and with Eqs. (22a–d) it follows

$$\rho \frac{\partial s}{\partial \mathbf{e}^e} = -\frac{\partial \boldsymbol{\tau}}{\partial T}, \quad \rho \frac{\partial s}{\partial \epsilon_v^e} = -\frac{\partial \sigma_m}{\partial T}, \quad (25a,b)$$

$$\rho \frac{\partial s}{\partial \mathbf{p}} = -\frac{\partial \mathbf{P}}{\partial T}, \quad \rho \frac{\partial s}{\partial \boldsymbol{\omega}} = -\frac{\partial \boldsymbol{\Omega}}{\partial T}. \quad (25c,d)$$

Thus, the entropy is closely related to the thermal dependence of the stresses.

Now, the reversible rate of heat is given by Eqs. (11), (24) and (25)

$$\rho \dot{q}_0 = -T \frac{\partial \boldsymbol{\tau}}{\partial T} \cdot \dot{\mathbf{e}}^e - T \frac{\partial \sigma_m}{\partial T} \dot{\epsilon}_v^e + \rho T \frac{\partial s}{\partial T} \dot{T} - T \frac{\partial \mathbf{P}}{\partial T} \cdot \dot{\mathbf{p}} - T \frac{\partial \boldsymbol{\Omega}}{\partial T} \cdot \dot{\boldsymbol{\omega}}. \quad (26)$$

The equations of the start of this section are the starting point for the development of constitutive laws in the remainder of this section.

3.1. Model for thermoelastic material

A thermoelastic material is a reversible model in the sense that $\dot{d} = 0$. One should however be careful when considering non-isothermal inhomogeneous deformation of a body of thermoelastic material. Such loading generally involves heat flow and is thus an *irreversible* process although the (mechanical) dissipation \dot{d} is zero, cf. the second term of Eq. (14).

For the thermoelastic model, $\mathbf{p} \equiv \boldsymbol{\omega} \equiv \mathbf{0}$ and the classical isotropic thermoelastic model reads

$$\boldsymbol{\tau} = 2G\mathbf{e}^e, \quad \sigma_m = K(\epsilon_v^e - \epsilon_v^T), \quad (27a,b)$$

where

$$\epsilon_v^T(T) = \alpha_v(T - T_r), \quad (27c)$$

is the *volumetric* thermal strain and T_r is a reference temperature. Moreover, G , K and α_v are the *constant* shear modulus, bulk modulus and coefficient of thermal expansion respectively.¹

Equations (27a–c) yield

$$\frac{\partial \boldsymbol{\tau}}{\partial T} = \mathbf{0}, \quad \frac{\partial \sigma_m}{\partial T} = -K\alpha_v, \quad (28a, b)$$

and with Eqs. (25a,b) we conclude that the entropy is independent of the deviatoric part of the elastic strain. The dependence of the final state variable, T , is given by the heat capacity. Since $\dot{d} = 0$, the rate of heat equals the reversible rate of heat which, in turn, is given by the rate of entropy. Thus, with Eqs. (11), (26) and (28a,b)

$$\rho \dot{q} = TK\alpha_v \dot{\epsilon}_v^e + \rho T \frac{\partial s}{\partial T} \dot{T}. \quad (29)$$

From the last part we identify the heat capacity

$$c_v^e \equiv T \frac{\partial s}{\partial T}. \quad (30)$$

Now, since s is independent of \mathbf{e}^e , c_v^e also must be independent of the deviatoric part of the elastic strain. Furthermore, it is also independent of ϵ_v^e . To see this, formulate

$$\rho \frac{\partial c_v^e}{\partial \epsilon_v^e} = \frac{\partial}{\partial \epsilon_v^e} \left(\rho T \frac{\partial s}{\partial T} \right) = T \frac{\partial}{\partial T} \left(\rho \frac{\partial s}{\partial \epsilon_v^e} \right) = -T \frac{\partial}{\partial T} (K\alpha_v) = 0, \quad (31)$$

where Eqs. (25b) and (28b) are used. Thus, at most $c_v^e = c_v^e(T)$.

We now have all derivatives of the entropy, cf. Eqs. (25a,b) and (30). These equations can be integrated with Eqs. (28a,b). To distinguish this form of the entropy from the subsequent forms, we denote it s^e . With the arbitrary integration constant determined from $s^e(0, T_r) = 0$, integration yields

$$\rho s^e(\epsilon_v^e, T) = K\alpha_v \epsilon_v^e + \rho \int_{T_r}^T \frac{c_v^e(\theta)}{\theta} d\theta. \quad (32)$$

All relevant derivatives of f are now formulated, cf. Eqs. (20) and (22a,b). To distinguish this form of the free energy from the subsequent forms it is denoted f^e . Integration yields

¹ Defined in this way, the coefficient of thermal expansion equals three times the conventional *linear* coefficient of thermal expansion.

$$\rho f^e(\mathbf{e}^e, \epsilon_v^e, T) = G \mathbf{e}^e \cdot \mathbf{e}^e + \frac{1}{2} K \epsilon_v^e (\epsilon_v^e - 2\epsilon_v^T) - \rho \int_{T_r}^T \frac{T - \theta}{\theta} c_v^e(\theta) d\theta, \quad (33)$$

where the arbitrary integration constant is chosen by $f^e(\mathbf{0}, 0, T_r) = 0$. The middle term might need some extra attention; it might appear more intuitive to get $K(\epsilon_v^e - \epsilon_v^T)^2/2$ which gives no contribution to the free energy if the volumetric thermal strain equals the volumetric elastic strain, or equivalently, if $\sigma_m = 0$. This form is indeed found in some formulations. As is easily verified, it is in error.

3.2. Framework for models of thermoelastoplastic material

With classical models of plasticity, it is assumed that the material reacts as the elastic, virgin material if the material is unloaded. Thus, effects of e.g. elastic spring back of dislocation loops and formation of voids are neglected. The assumption is usually used only in its mechanical sense, i.e. in terms of stress and strain. This is however unsatisfactorily in a thermodynamic setting. For instance, in the often assumed isothermal case, the energy conversion is dominated by the rate of heat. To see this, study the results of the previous section. The heat, Eq. (29), is linear in the elastic strain while the work, Eq. (21), is quadratic. Thus, the energy conversion in an isothermal elastic loading is dominated by the heat.

We will here study the effects of extending the mechanical assumption above to its thermal counterpart. Thus, we will assume that the material acts, both mechanically and thermally, as the virgin elastic material if the state of plastic rearrangement, modelled with \mathbf{p} , is constant. (Throughout this section, ω is set to zero.)

For a moment, assume that the state is given by the alternative set of state variables, τ , σ_m , T , and \mathbf{p} . For the model to react mechanically as the virgin elastoplastic model when $\dot{\mathbf{p}} = \mathbf{0}$ the following relations shall hold:

$$\frac{\partial e_{(ij)}}{\partial \tau_{(ij)}} = \frac{1}{2G}, \quad \frac{\partial e_{ij}}{\partial \sigma_m} = 0, \quad \frac{\partial e_{ij}}{\partial T} = 0, \quad (34a,b,c)$$

$$\frac{\partial \epsilon_v}{\partial \tau_{ij}} = 0, \quad \frac{\partial \epsilon_v}{\partial \sigma_m} = \frac{1}{K}, \quad \frac{\partial \epsilon_v}{\partial T} = \alpha_v, \quad (34d,e,f)$$

independent of the level of \mathbf{p} . (No summation shall be performed on repeated indices in parentheses.) Integration yields

$$\mathbf{e}(\tau, \mathbf{p}) = \mathbf{e}^e(\tau) + \mathbf{e}^p(\mathbf{p}), \quad \epsilon_v(\sigma_m, T, \mathbf{p}) = \epsilon_v^e(\sigma_m, T) + \epsilon_v^p(\mathbf{p}). \quad (35a,b)$$

Thus, the strain splits up in two parts; one thermoelastic, which equals the one for the pure thermoelastic material of the previous section, and one plastic part which is independent of stress and temperature. This result, previously noted by Kestin and Rice (1970), is a result of using constant values of K , G and α_v and assuming that, at a reference state with $\sigma = \mathbf{0}$ and $T = T_r$, the values of ϵ and \mathbf{p} vanish.

It is now convenient to define two “kinematic” relations between the internal state variable \mathbf{p} and the plastic strain, viz.

$$\dot{\mathbf{e}}^p \equiv \mathbf{H}^e(\mathbf{p}) \cdot \dot{\mathbf{p}}, \quad \dot{\epsilon}_v^p \equiv \mathbf{H}^e(\mathbf{p}) \cdot \dot{\mathbf{p}}. \quad (36a,b)$$

Due to the unique relation between σ , ϵ^e and T , given by Eqs. (27a–c), the state may equally be characterized by the state variables \mathbf{e}^e , ϵ_v^e , T , and \mathbf{p} . With these state variables, the rate of dissipation is given by Eq. (23) with Eqs. (35) and (36a,b)

$$\rho \dot{d} = \Pi \cdot \dot{\mathbf{p}}, \quad (37a)$$

where

$$\mathbf{\Pi} \equiv \boldsymbol{\tau} \cdot \mathbf{H}^e + \sigma_m \mathbf{H}^e - \mathbf{P}, \quad (37b)$$

defines a “plastic dissipation stress”. A method to develop thermodynamically consistent models, is to assume the existence of a dissipation potential F , convex in $\mathbf{\Pi}$, from which the rate of \mathbf{p} is calculated. That is, with $\dot{\lambda} \geq 0$, $\dot{\mathbf{p}} = \dot{\lambda} \partial F / \partial \mathbf{\Pi}$, cf. Generalized Standard Materials (Nguyen, 2000).

Now, partition the free energy and the entropy according to

$$f = f^e + f^p, \quad s = s^e + s^p, \quad (38a,b)$$

where s^e and f^e are given by Eqs. (32) and (33) respectively. We will now show that s^p is, at most, a function of \mathbf{p} and that f^p is, at most, a function of T and \mathbf{p} . To show this, note that if the model is to degenerate into the thermoelastic model in the manner described above, \dot{q} must, for $\dot{\mathbf{p}} = \mathbf{0}$, be identical to the rate of heat of the thermoelastic model. As shown in Eq. (37a), $\dot{\mathbf{p}} = \mathbf{0}$ gives $\dot{d} = 0$, thus, by Eq. (13) it suffices to assure that for $\dot{\mathbf{p}} = \mathbf{0}$, \dot{q}_0 equals the rate of heat of the thermoelastic model. With Eqs. (11), (29), (30) and (38b)

$$(\rho \dot{q}_0) \rho T \frac{\partial(s^e + s^p)}{\partial \mathbf{e}^e} \cdot \dot{\mathbf{e}}^e + \rho T \frac{\partial(s^e + s^p)}{\partial \epsilon_v^e} \dot{\epsilon}_v^e + \rho T \frac{\partial(s^e + s^p)}{\partial T} \dot{T} = TK \alpha_v \dot{\epsilon}_v^e + \rho c_v^e \dot{T}. \quad (39)$$

The equality holds if the plastic part of the entropy is, at most, a function of \mathbf{p} , i.e. $s^p = s^p(\mathbf{p})$. In order to study the possible forms of f^p , establish

$$\rho \dot{f}^p = \mathbf{P} \cdot \dot{\mathbf{p}} - \rho s^p \dot{T}, \quad (40)$$

where Eqs. (17), (21) and (38b) have been used. This equation immediately shows that, at most, $f^p = f^p(\mathbf{p}, T)$. The possible forms of the plastic stress can now be examined. By the results above and Eq. (22c), at most, $\mathbf{P} = \mathbf{P}(\mathbf{p}, T)$. Moreover, by the results above and Eq. (25c), \mathbf{P} is, at most, linearly dependent on the temperature. Thus, the most general form is given by

$$\mathbf{P} = \mathbf{B}^p(\mathbf{p}) - \mathbf{b}^p(\mathbf{p})(T - T_r), \quad (41)$$

where the functions $\mathbf{B}^p(\mathbf{p})$ and $\mathbf{b}^p(\mathbf{p})$ have to be determined by comparisons with e.g. experiments or simulations.

There is a close connection between the plastic part of the entropy and the thermal dependence of the plastic stress. To see this, formulate

$$\rho \frac{\partial s^p}{\partial \mathbf{p}} = - \frac{\partial \mathbf{P}}{\partial T} = \mathbf{b}^p, \quad (42)$$

where Eqs. (25c), (38b) and (41) have been used. Integration gives

$$\rho s^p(\mathbf{p}) = \int_0^{\mathbf{p}} \mathbf{b}^p(\hat{\mathbf{p}}) d\hat{\mathbf{p}}, \quad (43)$$

where the integration constant is chosen by $s^p(\mathbf{0}) = 0$. For completeness, the rate of heat is written

$$\rho \dot{q} = TK \alpha_v \dot{\epsilon}_v^e + \rho c_v^e \dot{T} + T \mathbf{b}^p \cdot \dot{\mathbf{p}} - \mathbf{\Pi} \cdot \dot{\mathbf{p}} \quad (44)$$

where Eqs. (13), (26), (27) and (37a) have been used. Thus, the present assumptions yield a heat capacity unaffected by plastic strain. Bever et al. (1973) reviews a large number of experiments where the heat capacity is measured for annealed and heavily plastically deformed metallic specimens. They conclude that the difference in heat capacity is always less than 1% and often less than the sensitivity of the recording technique. Thus, the result of the present assumptions is in accordance with experiments.

An interesting deduction from Eqs. (41) and (43) is that, with temperature independent plastic stress, plastic deformation is not associated with a permanent change of the entropy. The fact that a model is unable to produce a permanent change of entropy is not necessarily a reason to reject the model. However,

it limits the modelling capabilities. In order to elaborate, consider an isothermal loading sequence where the stress is varied from zero to a maximum and then back to zero again and producing plastic deformation, cf. Fig. 3. With $\mathbf{b}^p \equiv \mathbf{0}$, both Δs^p and Δs^e equal zero and thus $\Delta s = 0$. In Eq. (44) only the last term, equal to the rate of dissipation contributes to the heat. Thus, all dissipated energy is “given away” to the environment. With $\mathbf{b}^p \neq \mathbf{0}$, the dissipation does not equal the heat.

The plastic part of the free energy is now derived by integrating Eqs. (20) and (22c) with Eqs. (41) and (43). The result is

$$\rho f^p(T, \mathbf{p}) = \int_0^{\mathbf{p}} \mathbf{B}^p(\hat{\mathbf{p}}) d\hat{\mathbf{p}} - (T - T_r) \int_0^{\mathbf{p}} \mathbf{b}^p(\hat{\mathbf{p}}) d\hat{\mathbf{p}}, \quad (45)$$

where the integration constant is set by choosing $f^p(T_r, \mathbf{0}) = 0$.

We conclude this section by noting that with $\dot{\mathbf{p}} = \mathbf{0}$, $\dot{\epsilon} = \dot{\epsilon}^e$ by mechanical degeneration and $\dot{s} = \dot{s}^e$ by thermal degeneration. However, for this case, $\dot{f} \neq \dot{f}^e$ in general. This follows by the form of the rate of plastic free energy, cf. Eq. (40), which shows that, for this case, f^p varies linearly with the temperature and $\dot{f} \neq \dot{f}^e$.

3.3. Framework for material models of damaged material

Creation, growth and coalescence of microscopic cracks and voids are frequently identified in the fracture process of solids. At very high temperatures, above about half the homologous melting temperature of metals, grain boundary voids form. This mechanism is identified as a major life limiting mechanism for e.g. turbine blades in high-performance jet engines. At moderate temperatures, plastic deformation induces voids at notches and crack tips in ductile solids. These voids coalesce and form a crack surface, thus, limiting the load bearing capacity. Microscopic cracks are also identified as a major mechanism for fracture by e.g. fatigue in brittle composites. Microscopic cracks and voids may also heal by subjecting the material to compressive stress, high temperature or a combination. Sintering is indeed used as a life-prolonging process for military jet-engines. Thus, a general theory for CDM should facilitate modelling of the effects of the processes of nucleation, growth, coalescence and healing of microscopic cracks and voids by stress, plasticity and thermally driven mechanisms.

Generalizing the original ideas of Kachanov (1958), an effective symmetric stress tensor, $\tilde{\sigma}$, is defined to be given by a linear transformation of the homogenized stress tensor,

$$\sigma_{ij} = N_{ijkl}^{\sigma}(\omega) \tilde{\sigma}_{kl}. \quad (46)$$

Here, the fourth-order integrity tensor \mathbf{N}^{σ} is a generalization of the Kachanov integrity function $\psi = 1 - \omega$, cf. Eq. (1). The integrity is defined to be a function of the damage variables only, thus $\mathbf{N}^{\sigma} = \mathbf{N}^{\sigma}(\omega)$. The interpretation of the integrity tensor is immediate: if some components of \mathbf{N}^{σ} equal zero, the material point has lost its capacity to carry load in a certain direction; e.g., if $N_{11kl}^{\sigma} = 0$ the model has lost its capacity to support load in the x_1 -direction. For a virgin material, $\omega = \mathbf{0}$ and the effective stress equals the homogenized stress, thus $N_{ijkl}^{\sigma}(\mathbf{0}) = (\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})/2$. During a loading history, the components of ω may vary, both increase; known as a deteriorating process and decrease; known as a healing process. It is reasonable to assume that the components of \mathbf{N}^{σ} are monotonically decreasing functions of the components of ω , although this condition is not explicitly applied in the further development.

For models exhibiting isotropic behaviour, \mathbf{N}^{σ} must be an isotropic fourth-order tensor. Generally, the Cartesian components of an isotropic fourth-order tensor is given by $N_{ijkl}^{\sigma} = c\delta_{ij}\delta_{kl} + d\delta_{ik}\delta_{jl} + e\delta_{il}\delta_{jk}$ where $c(\omega)$, $d(\omega)$ and $e(\omega)$ are three independent scalar functions. By use of the symmetries of the homogenized and effective stress tensors, \mathbf{N}^{σ} can be replaced by two scalar integrity functions. Thus,

$$\tau_{ij} = N^G(\omega) \tilde{\tau}_{ij}, \quad \sigma_m = N^K(\omega) \tilde{\sigma}_m. \quad (47a,b)$$

Hence, at most *two* independent scalar integrity functions can be introduced in an isotropic model. This conclusion can be compared to the formulation of Cauvin and Testa (1999) who conclude that only *two scalar damage variables* can be used in an isotropic theory. Thus, the present formulation opens up for a more general formulation.

In line with Eqs. (47a,b), the effective plastic stress $\tilde{\mathbf{P}}$ associated with the damaged plastic state of the material is introduced similarly as the effective stress,

$$\mathbf{P} = \mathbf{N}^{\mathbf{P}}(\omega) \tilde{\mathbf{P}}, \quad (48)$$

where $\mathbf{N}^{\mathbf{P}}$ is a (collection of) isotropic tensor(s) defined to depend on ω only. For a virgin material, $\omega = \mathbf{0}$ and $\mathbf{P} = \mathbf{N}^{\mathbf{P}}(\mathbf{0}) \tilde{\mathbf{P}} = \tilde{\mathbf{P}}$.

We will now generalize the principle of strain equivalence similarly as is done for plasticity above. Thus, we state:

If we set $\dot{\omega} = \mathbf{0}$ and replace all effective stresses by their homogenized counterparts, the model shall degenerate both mechanically and thermally to the corresponding thermoelastoplastic model.

With the present integrity functions, this principle of mechanical and thermal equivalence is equivalent to requiring that the damaged model shall act in the same way as the thermoelastoplastic model if we set $\dot{\omega} = \omega = \mathbf{0}$.

With $\omega \neq \mathbf{0}$, the material is expected to be “weaker” than the undamaged material. To allow for this, introduce material parameters and functions that may be influenced by ω , viz. $\tilde{G}(\omega)$, $\tilde{K}(\omega)$, $\tilde{\alpha}_v(\omega)$, $\tilde{\mathbf{H}}^e(\mathbf{p}, \omega)$ and $\tilde{\mathbf{H}}^\epsilon(\mathbf{p}, \omega)$. A tilde over a symbol is here introduced to distinguish the present functions from the parameters and functions used in the previous sections. The relations between these functions and parameters are presently explored.

For a moment, assume that the state is given by the alternative set of state variables, τ , σ_m , T , \mathbf{p} and ω . For the model to react according to the mechanical aspects of the principle, the following relations shall hold,

$$\frac{\partial e_{(ij)}}{\partial \tau_{(ij)}} = \frac{1}{2\tilde{G}}, \quad \frac{\partial e_{ij}}{\partial \sigma_m} = 0, \quad \frac{\partial e_{ij}}{\partial T} = 0, \quad \frac{\partial \mathbf{e}}{\partial \mathbf{p}} = \tilde{\mathbf{H}}^e, \quad (49a,b,c,d)$$

$$\frac{\partial \epsilon_v}{\partial \tau_{ij}} = 0, \quad \frac{\partial \epsilon_v}{\partial \sigma_m} = \frac{1}{\tilde{K}}, \quad \frac{\partial \epsilon_v}{\partial T} = \tilde{\alpha}_v, \quad \frac{\partial \epsilon_v}{\partial \mathbf{p}} = \tilde{\mathbf{H}}^\epsilon. \quad (49e,f,g,h)$$

Integration of these equations yields

$$\mathbf{e}(\tau, \mathbf{p}, \omega) = \tilde{\mathbf{e}}^e(\tau, \omega) + \tilde{\mathbf{e}}^p(\mathbf{p}, \omega) + \tilde{\mathbf{e}}^\omega(\omega), \quad (50a)$$

$$\epsilon_v(\sigma_m, T, \mathbf{p}, \omega) = \tilde{\epsilon}_v^e(\sigma_m, T, \omega) + \tilde{\epsilon}_v^p(\mathbf{p}, \omega) + \tilde{\epsilon}_v^\omega(\omega), \quad (50b)$$

where

$$\tilde{\mathbf{e}}^e = \frac{\tau}{2\tilde{G}}, \quad \tilde{\mathbf{e}}^p = \int_0^{\mathbf{p}} \tilde{\mathbf{H}}^e(\hat{\mathbf{p}}, \omega) d\hat{\mathbf{p}}, \quad (51a,b)$$

$$\tilde{\epsilon}_v^e = \frac{\sigma_m}{\tilde{K}} + \tilde{\alpha}_v(T - T_r), \quad \tilde{\epsilon}_v^p = \int_0^{\mathbf{p}} \tilde{\mathbf{H}}^\epsilon(\hat{\mathbf{p}}, \omega) d\hat{\mathbf{p}}. \quad (51c,d)$$

It can be noted that, as for the thermoelastoplastic model, the strain is split into parts. Now, introduce the effective stresses Eqs. (47a,b) into Eqs. (51a,c)

$$\tilde{\mathbf{e}}^e = \frac{N^G \tilde{\boldsymbol{\tau}}}{2\tilde{G}}, \quad \tilde{\epsilon}_v^e = \frac{N^K \tilde{\sigma}_m}{\tilde{K}} + \tilde{\alpha}_v (T - T_r). \quad (52a,b)$$

According to the principle, these strain components shall equal the elastic strain components of Eqs. (27a–c) if the effective stresses are replaced by the homogenized stresses. This is achieved if

$$\tilde{G} = N^G G, \quad \tilde{K} = N^K K, \quad \tilde{\alpha}_v = \alpha_v. \quad (53a,b,c)$$

Furthermore, according to the principle, the plastic strain components shall equal the plastic strain components of Eqs. (36a,b). This is achieved if $\tilde{\mathbf{H}}^e$ and $\tilde{\mathbf{H}}^e$ are independent of $\boldsymbol{\omega}$ and given by the corresponding functions of the undamaged model, viz.

$$\tilde{\mathbf{H}}^e = \mathbf{H}^e(\mathbf{p}), \quad \tilde{\mathbf{H}}^e = \mathbf{H}^e(\mathbf{p}). \quad (54a,b)$$

Finally, the principle does not allow for a “damage strain”. Thus,

$$\tilde{\mathbf{e}}^\omega \equiv \mathbf{0}, \quad \tilde{\epsilon}_v^\omega \equiv 0. \quad (55a,b)$$

To sum up, the strain of the damaged material model is split according to

$$\mathbf{e} = \mathbf{e}^e\left(\frac{\boldsymbol{\tau}}{N^G}\right) + \mathbf{e}^p(\mathbf{p}), \quad \epsilon_v = \epsilon_v^e\left(\frac{\sigma_m}{N^K}, T\right) + \epsilon_v^p(\mathbf{p}), \quad (56a,b)$$

where the functions $\mathbf{e}^e(\boldsymbol{\tau})$ and $\epsilon_v^e(\sigma_m, T)$ are given by Eqs. (27a–c) and the functions $\mathbf{e}^p(\mathbf{p})$ and $\epsilon_v^p(\mathbf{p})$ by integration of Eqs. (36a,b). Moreover, the principle is followed if the plastic stress is replaced with the effective plastic stress in the constitutive relation for the undamaged model, cf. Eq. (41), viz.

$$\tilde{\mathbf{P}} = \mathbf{B}^p(\mathbf{p}) - \mathbf{b}^p(\mathbf{p})(T - T_r). \quad (57)$$

As indicated in Eqs. (56a,b) there is a unique relation between $\boldsymbol{\sigma}$, ϵ^e , T and $\boldsymbol{\omega}$. The state can then be characterised by the original set of state variables: \mathbf{e}^e , ϵ_v^e , T , \mathbf{p} , and $\boldsymbol{\omega}$. With these state variables, the rate of dissipation is given by Eq. (23) with (36a,b), (37b) and (56a,b)

$$\rho \dot{d} = \boldsymbol{\Pi} \cdot \dot{\mathbf{p}} - \boldsymbol{\Omega} \cdot \dot{\boldsymbol{\omega}}, \quad (58)$$

where $\boldsymbol{\Pi}$ is given by Eq. (37b). Thus, a plasticity model, with positive dissipation, is extended to a damage model with positive dissipation, by requiring $-\boldsymbol{\Omega} \cdot \dot{\boldsymbol{\omega}} \geq 0$ for the damage model. If we assume the existence of a dissipation potential M , convex in $\boldsymbol{\omega}$, non-negative dissipation associated with changes in $\boldsymbol{\omega}$ is guaranteed if $\dot{\boldsymbol{\omega}} = -\dot{\mu} \partial M / \partial \boldsymbol{\Omega}$ with $\dot{\mu} \geq 0$.

To conform with the thermal aspects of the principle, the rate of heat of the damage model shall equal the rate of heat of the undamaged model, Eq. (44), if $\dot{\boldsymbol{\omega}} = \boldsymbol{\omega} = \mathbf{0}$. Since, the rate of dissipation, in this case, equals the rate of dissipation for the thermoelastoplastic model, it suffices that the *reversible* rates of heat are equal, cf. Eq. (13). Using the effective stresses, Eqs. (47a,b) and (48), the reversible rate of heat, Eq. (26), is written

$$\rho \dot{q}_0 = -TN^G \frac{\partial \tilde{\boldsymbol{\tau}}}{\partial T} \cdot \dot{\mathbf{e}}^e - TN^K \frac{\partial \tilde{\sigma}_m}{\partial T} \dot{\epsilon}_v^e - TN^p \frac{\partial \tilde{\mathbf{P}}}{\partial T} \cdot \dot{\mathbf{p}} - T \frac{\partial \boldsymbol{\Omega}}{\partial T} \cdot \dot{\boldsymbol{\omega}} + \rho T \frac{\partial s}{\partial T} \dot{T}. \quad (59)$$

Using the constitutive relations, Eqs. (27a–c), (56a,b) and (57), this equation yields

$$\rho \dot{q}_0 = TN^K K \alpha_v \dot{\epsilon}_v^e + TN^p \mathbf{b}^p \cdot \dot{\mathbf{p}} - T \frac{\partial \boldsymbol{\Omega}}{\partial T} \cdot \dot{\boldsymbol{\omega}} + \rho T \frac{\partial s}{\partial T} \dot{T}. \quad (60)$$

Applying the principle and putting $\dot{\boldsymbol{\omega}} = \boldsymbol{\omega} = \mathbf{0}$ yields

$$\rho \dot{q}_0|_{\dot{\boldsymbol{\omega}}=\boldsymbol{\omega}=\mathbf{0}} = TK \alpha_v \dot{\epsilon}_v^e + T \mathbf{b}^p \cdot \dot{\mathbf{p}} + \rho T \frac{\partial s}{\partial T} \bigg|_{\boldsymbol{\omega}=\mathbf{0}} \dot{T}, \quad (61)$$

which conforms with the principle, cf. the three first terms of Eq. (44), if

$$T \frac{\partial s}{\partial T} \bigg|_{\omega=0} = c_v^e. \quad (62)$$

This equation allows for a damage dependent heat capacity, $\tilde{c}(T, \omega)$. In principle, the part of the material point which has undergone damage $(1 - N^K)$ contribute to the total heat capacity with its heat capacity at constant pressure, i.e. c_p^e . The remaining part of the material point (N^K) contribute with the heat capacity at constant volume. That is,

$$\rho \tilde{c} = (1 - N^K) \rho c_p^e + N^K \rho c_v^e = \rho c_v^e + (1 - N^K) T K \alpha_v^2, \quad (63)$$

where we have used $c_p^e - c_v^e = T K \alpha_v^2 / \rho$. Equation (62), for an arbitrary level of damage, thus takes the form

$$\rho T \frac{\partial s}{\partial T} = \rho c_v^e + (1 - N^K) T K \alpha_v^2. \quad (64)$$

Equations (25a–d) with Eqs. (27a–c), (56a,b) and (57) give

$$\rho \frac{\partial s}{\partial \epsilon^e} = \mathbf{0}, \quad \rho \frac{\partial s}{\partial \epsilon_v^e} = N^K K \alpha_v, \quad (65a,b)$$

$$\rho \frac{\partial s}{\partial \mathbf{p}} = N^P \mathbf{b}^P, \quad \rho \frac{\partial s}{\partial \omega} = -\frac{\partial \Omega}{\partial T}, \quad (65c,d)$$

which, together with Eq. (64), yield a system of partial differential equations for the entropy. We immediately recognize that the entropy is independent of the deviatoric part of the elastic strain. Integration of Eqs. (64) and (65b,c) gives

$$s(\epsilon_v^e, T, \mathbf{p}, \omega) = s^{\text{ep}}(\epsilon_v^e, T, \mathbf{p}, \omega) + s^c(\omega), \quad (66a)$$

where

$$\rho s^{\text{ep}}(\epsilon_v^e, T, \mathbf{p}, \omega) = N^K K \alpha_v (\epsilon_v^e - \epsilon_v^T) + \rho \int_{T_r}^T \frac{c_v^e(\theta)}{\theta} d\theta + K \alpha_v \epsilon_v^T + N^P \int_0^P \mathbf{b}^P(\hat{\mathbf{p}}) d\hat{\mathbf{p}}. \quad (66b)$$

The function s^{ep} is associated with the level of stress, temperature and plastic deformation. It is thus denoted the elastoplastic part of the entropy. The second function s^c is an, as yet, unspecified function of damage. It is here interpreted as corresponding to the cohesive entropy of microscopic cracks and voids. The split of the entropy in two parts according to Eq. (66a) suggests a similar partitioning of the corresponding damage stress, cf. Eqs. (22d) and (25d), in an elastoplastic and a cohesive part according to

$$\Omega \equiv -\Omega^{\text{ep}} + \Omega^c, \quad (67a)$$

where

$$\frac{\partial \Omega^{\text{ep}}}{\partial T} = \rho \frac{\partial s^{\text{ep}}}{\partial \omega} = \frac{dN^K}{d\omega} K \alpha_v (\epsilon_v^e - \epsilon_v^T) + \frac{dN^P}{d\omega} \int_0^P \mathbf{b}^P(\hat{\mathbf{p}}) d\hat{\mathbf{p}}, \quad (67b)$$

and

$$\frac{\partial \Omega^c}{\partial T} = -\rho \frac{\partial s^c(\omega)}{\partial \omega}. \quad (67c)$$

Thus, the cohesive damage stress, Ω^c , is at most a linear function of the temperature. The unconventional sign for the elastoplastic damage stress, Ω^{ep} , in Eq. (67a) is chosen in order to get an essentially positive

value of $\mathbf{\Omega}^{\text{ep}}$. To elucidate the resemblance with the plastic stress, cf. Eq. (41), we recognize that $\mathbf{\Omega}^{\text{c}} = \mathbf{\Omega}^{\text{c}}(T, \omega)$ and write

$$\mathbf{\Omega}^{\text{c}} = \mathbf{B}^{\text{c}}(\omega) - \mathbf{b}^{\text{c}}(\omega)(T - T_r). \quad (68)$$

The new functions \mathbf{B}^{c} and \mathbf{b}^{c} have to be determined by comparisons with, for example, experiments or simulations. Differentiation and integration according to Eq. (67c) gives

$$\rho s^{\text{c}} = \int_0^\omega \mathbf{b}^{\text{c}}(\hat{\omega}) d\hat{\omega}. \quad (69)$$

For completeness, the rate of heat is written

$$\begin{aligned} \rho \dot{q} = & TN^{\text{K}} K \alpha_v \dot{\epsilon}_v^{\text{c}} + \rho \tilde{c} \dot{T} + (TN^{\text{P}} \mathbf{b}^{\text{p}} - \mathbf{\Pi}) \cdot \dot{\mathbf{p}} \\ & + \left[T \frac{dN^{\text{K}}}{d\omega} K \alpha_v (\epsilon_v^{\text{c}} - \epsilon_v^{\text{T}}) + T \frac{dN^{\text{P}}}{d\omega} \int_0^{\text{p}} \mathbf{b}^{\text{p}}(\hat{\mathbf{p}}) d\hat{\mathbf{p}} + T \mathbf{b}^{\text{c}} + \mathbf{\Omega} \right] \cdot \dot{\omega}, \end{aligned} \quad (70)$$

where Eqs. (13), (60), (64), (67a–c) and (68) have been used.

All partial derivatives of the free energy are now given by Eqs. (20) and (22a–d) with Eqs. (27a–c), (47a,b), (48), (56a,b), (57), (65a) and (68). Integration yields

$$f(\mathbf{e}^{\text{e}}, \epsilon_v^{\text{c}}, T, \mathbf{p}, \omega) = f^{\text{ep}}(\mathbf{e}^{\text{e}}, \epsilon_v^{\text{c}}, T, \mathbf{p}, \omega) + f^{\text{c}}(T, \omega), \quad (71a)$$

where

$$\begin{aligned} \rho f^{\text{ep}} = & N^{\text{G}} G \mathbf{e}^{\text{e}} \cdot \mathbf{e}^{\text{e}} + \frac{1}{2} N^{\text{K}} K (\epsilon_v^{\text{c}} - \epsilon_v^{\text{T}})^2 - \rho \int_{T_r}^T \frac{T - \theta}{\theta} c_v^{\text{c}}(\theta) d\theta - \frac{1}{2} K (\epsilon_v^{\text{T}})^2 \\ & + N^{\text{P}} \left\{ \int_0^{\text{p}} \mathbf{B}^{\text{p}}(\hat{\mathbf{p}}) d\hat{\mathbf{p}} - (T - T_r) \int_0^{\text{p}} \mathbf{b}^{\text{p}}(\hat{\mathbf{p}}) d\hat{\mathbf{p}} \right\}, \end{aligned} \quad (71b)$$

and

$$\rho f^{\text{c}} = \int_0^\omega \mathbf{B}^{\text{c}}(\hat{\omega}) d\hat{\omega} - (T - T_r) \int_0^\omega \mathbf{b}^{\text{c}}(\hat{\omega}) d\hat{\omega}, \quad (71c)$$

where the arbitrary integration constant is set by requiring $f(\mathbf{0}, 0, T_r, \mathbf{0}, \mathbf{0}) = 0$. We can now compare the *derived* version of the free energy, Eqs. (71a–c), and the *ad hoc* versions, Eqs. (2) and (3). The most obvious difference is the new term f^{c} . This term opens up new possibilities in CDM, namely to model healing in an integrated manner. This is demonstrated in one of the examples in the next section. The term, f^{ep} , yields the effect of temperature on the free energy; this effect is not present in Eqs. (2) and (3). Moreover, Eqs. (71a–c) yield the *specific* and *possible* form of the effects of elastic strain, temperature, plastic and damage internal-variables consistent with assumptions of the behaviour of the material. It can easily be seen that both Eqs. (2) and (3) can be retained with suitable choices of integrity functions.

The elastoplastic part of the damage stress is finally given by differentiation of Eq. (71b) according to Eqs. (22d) and (67a).

$$\mathbf{\Omega}^{\text{ep}} = - \frac{dN^{\text{G}}}{d\omega} G \mathbf{e}^{\text{e}} \cdot \mathbf{e}^{\text{e}} - \frac{1}{2} \frac{dN^{\text{K}}}{d\omega} K (\epsilon_v^{\text{c}} - \epsilon_v^{\text{T}})^2 - \frac{dN^{\text{P}}}{d\omega} \left\{ \int_0^{\text{p}} \mathbf{B}^{\text{p}}(\hat{\mathbf{p}}) d\hat{\mathbf{p}} - (T - T_r) \int_0^{\text{p}} \mathbf{b}^{\text{p}}(\hat{\mathbf{p}}) d\hat{\mathbf{p}} \right\}. \quad (72)$$

This ends the derivation of a thermodynamically consistent framework for development of constitutive laws. Some capabilities of the framework will be demonstrated in Section 4.

4. Examples

In this section, the general relations derived above are used to demonstrate some of the capabilities of the framework for developing constitutive laws. The examples should be regarded as simplified illustrations and not full-fledged constitutive models. In order to emphasize the *structured* method to develop constitutive models, both examples start from the same model for a thermoelastic material. In the first example, this model is modified to introduce plasticity and in the second example, the thermoelastic model is modified to model creep damage. We will simplify as much as possible and limit the example to a uniaxial theory. Thus, with customary notation

$$\sigma = E(\epsilon^e - \epsilon^T), \quad \text{where } \epsilon^T = \alpha(T - T_r), \quad (73a,b)$$

governs the *elastic behaviour*. With a *constant* heat capacity, c , the elastic parts of the entropy and free energy are given by

$$\rho s^e = E\alpha\epsilon^e + \rho c \ln \frac{T}{T_r}, \quad (74a)$$

$$\rho f^e = \frac{1}{2} E\epsilon^e(\epsilon^e - 2\epsilon^T) - \rho c \left[T \ln \frac{T}{T_r} - (T - T_r) \right]. \quad (74b)$$

Moreover, the rate of heat, Eqs. (29) and (30), is given by

$$\rho \dot{q} = TE\alpha\dot{\epsilon}^e + \rho c \dot{T}. \quad (75)$$

In an *isothermal process* ($\dot{T} \equiv 0$), the rate of heat is thus, $\rho \dot{q} = TE\alpha\dot{\epsilon}^e$ and, in most cases (with $\alpha > 0$), heat has to be supplied proportionally with the elastic strain to keep the temperature constant. Moreover, in a finitely sized body, the process has to be “infinitely” slow in order for the temperature to be constant, cf. the discussion in the preamble to Section 3.1. In an *adiabatic process* ($\dot{q} \equiv 0$), Eq. (75) shows that the rate of temperature is proportional to the rate of elastic strain, viz. $\dot{T} = -TE\alpha\dot{\epsilon}^e/(\rho c)$. Thus, in most cases (with $\alpha > 0$), the temperature decreases with increasing elastic strain. It is interesting to study the difference in apparent elastic stiffness in an isothermal and adiabatic process. Equations (73a,b) yield the stress rate for an isothermal process while the same equations, with the the rate of temperature above, yield the stress rate for an adiabatic process, viz.

$$\dot{\epsilon}_{\text{iso}} = \frac{\dot{\sigma}}{E}, \quad \dot{\epsilon}_{\text{ad}} = \frac{\dot{\sigma}}{E_{\text{ad}}}, \quad (76a,b)$$

where $E_{\text{ad}} = E[1 + \alpha^2 ET/(\rho c)]$ is the “adiabatic” elastic modulus. Thus, the adiabatic process leads to a somewhat larger stiffness. In most cases, the difference is not large.

4.1. Elastoplastic model

Introducing *one scalar* internal variable for the plasticity, p , the plastic strain rate is given by a one-dimensional version of Eq. (36b). Here, the simplest possible form is chosen

$$\dot{\epsilon}^p = \dot{p}. \quad (77)$$

For simplicity, we choose the plastic stress, cf. Eq. (41), according to

$$P = E_p[p - \alpha_p(T - T_p)], \quad (78)$$

where E_p and α_p are constants. The plastic dissipation stress, Eq. (37b), reduces to

$$\Pi = \sigma - P, \quad (79)$$

We will now chose a yield criteria. In models of associated plasticity, this is the same as the dissipation potential. It shall therefore be a convex function of the plastic dissipation stress in order to secure positive dissipation, cf. Eq. (37a) and the discussion following it. A suitable yield criteria is

$$F(\Pi) \equiv |\Pi| - \sigma_Y = 0, \quad (80)$$

where σ_Y is a constant. With $E_p > 0$, *linear kinematic hardening* is modelled. With $E_p = 0$ *ideal plasticity* results. With $\alpha_p \neq 0$, a temperature dependent yield stress is accomplished. It is interesting to note that this is associated with a plastic part of the entropy. This is seen from Eq. (43), which in this case yields

$$\rho s^p = E_p \alpha_p p. \quad (81a)$$

The plastic part of the free energy, Eq. (45), reduces to

$$\rho f^p = \frac{1}{2} E_p p^2 - E_p \alpha_p (T - T_p) p. \quad (81b)$$

With the present dissipation potential, Eq. (80), the rate of the internal variable is

$$\dot{p} = \dot{\lambda} \frac{dF}{d\Pi} = \dot{\lambda} \text{sign}(\Pi), \quad (82)$$

where $\text{sign}(\Pi)$ denotes the sign of Π and $\dot{\lambda} \geq 0$. During plastic deformation, the yield condition, Eq. (80), must hold. Thus, $\dot{\sigma} = \dot{P}$, which with Eqs. (78) and (82) yields $\dot{\lambda} \text{sign}(\Pi) = \dot{\sigma}/E_p + \alpha_p \dot{T}$. Now, with Eqs. (77) and (82), the plastic strain rate is given by

$$\dot{\epsilon}^p = \frac{\dot{\sigma}}{E_p} + \alpha_p \dot{T}. \quad (83)$$

The total strain rate is given by the sum of the elastic and plastic strain rates Eqs. (35a,b). With Eqs. (73a,b) and (83) the strain rate is

$$\dot{\epsilon} = \frac{\dot{\sigma}}{E_T} + \alpha_T \dot{T}, \quad (84a)$$

where

$$\frac{1}{E_T} \equiv \frac{1}{E} + \frac{1}{E_p}, \quad \text{and} \quad \alpha_T \equiv \alpha + \alpha_p, \quad (84b,c)$$

are the elastoplastic tangent stiffness and coefficient of thermal expansion respectively. It may be noted that the plastic part of the entropy Eq. (81a) is associated with a “plastic” contribution to the thermal strain rate ($\alpha_p \dot{T}$). It should be noted that a thermally dependent plastic stress (P) inevitably results in a plastic part of the thermal strain rate ($\alpha_p \dot{T}$). This part appears unphysical.

For the present model, the rate of heat, Eq. (44), reduces to

$$\rho \dot{q} = TE \alpha \dot{\epsilon}^e + \rho c \dot{T} - [\text{sign}(\Pi) \sigma_Y - TE_p \alpha_p] \dot{\epsilon}^p. \quad (85)$$

For *isothermal elastoplastic deformation*, the strain rate is given by the first term in Eq. (84a)

$$\dot{\epsilon}_{\text{iso}} = \frac{\dot{\sigma}}{E_T}. \quad (86a)$$

For an *adiabatic elastoplastic deformation* the rate of the temperature can be calculated by setting Eq. (85) equal to zero. The result is

$$\dot{T} = \frac{T \alpha_T - \text{sign}(\Pi) \sigma_Y / E_p}{TE \alpha^2 + TE_p \alpha_p^2 - \text{sign}(\Pi) \sigma_Y \alpha_p + \rho c} \dot{\sigma}. \quad (86b)$$

The strain rate is then given from Eq. (84a) by

$$\dot{\epsilon}_{\text{ad}} = \frac{\dot{\sigma}}{E_{\text{T,ad}}}, \quad (86c)$$

where

$$\frac{1}{E_{\text{T,ad}}} = \frac{1}{E} + \frac{1}{E_p} - \frac{T\alpha_T - \text{sign}(\Pi)\sigma_Y/E_p}{TE\alpha^2 + TE_p\alpha_p^2 - \text{sign}(\Pi)\sigma_Y\alpha_p + \rho c}. \quad (86d)$$

Thus, the adiabatic tangent modulus can be smaller, larger or equal to the isothermal tangent modulus depending on the relative size of the different terms in Eq. (86d). Moreover, the tangent stiffness is not constant but changes due to the changing temperature.

4.2. Elastodamaged material

The elastic model is now modified to introduce *creep damage*. In order to simplify, no plasticity is considered, and only one scalar damage variable is introduced. Moreover, only isothermal processes are considered. With these limitations, the elastoplastic damage stress, Eq. (72), reduces to

$$\Omega^{\text{ep}} = \frac{1}{2}E\epsilon^2 = \frac{\tilde{\sigma}^2}{2E} = \frac{\sigma^2}{2E(1-\omega)^2}. \quad (87a)$$

The following form of the cohesive damage stress, Eq. (68), is considered

$$\Omega^c = E_c\omega, \quad (87b)$$

where, E_c is a constant. A convex damage surface is chosen as

$$M(\Omega) = \frac{\Omega^2}{2E} = \frac{(-\Omega^{\text{ep}} + \Omega^c)^2}{2E}, \quad (88)$$

where the normalization (the denominator $2E$) is chosen to give physically consistent dimensions. The damage rate is now calculated from the damage surface according to

$$\dot{\omega} = -\frac{dM}{d\Omega} = -\frac{\Omega}{E} = \frac{\sigma^2}{2E^2(1-\omega)^2} - \frac{E_c}{E}\omega, \quad (89)$$

which shows that both tensile and compressive stress produces an increase of damage. This is usually considered as non-physical. It can be remedied by introducing an additional condition for $\dot{\omega} > 0$, namely $\sigma \geq 0$. We will here only consider tensile stress. Moreover, Eq. (89) shows that σ must be larger than some criteria to produce $\dot{\omega} > 0$. If $\sigma > (E/2)\sqrt{E_c/E}$, $\dot{\omega}$ will always be positive for a constant stress.

Figure 4 shows the accumulation of damage and (elastic) strain during a “creep test” with a constant stress. Both damage and strain increase with an accelerating rate until fracture is predicted at $t_R = 1292$.

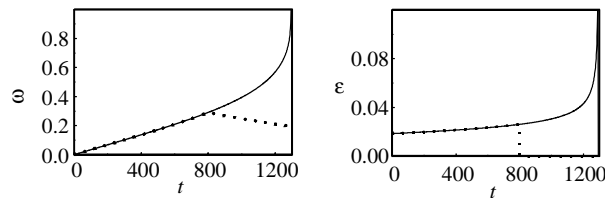


Fig. 4. Accumulation of creep damage (left) and strain (right) during a creep test using the following data: $E = 1000$, $E_c = 0.8$, $\sigma = 18.5$. The time to creep rupture is $t_R = 1292$ (solid lines). The dashed lines corresponds to an interrupted creep test where the specimen is unloaded at the time $t = 800$. The strain immediately decrease to zero and the damage slowly heals.

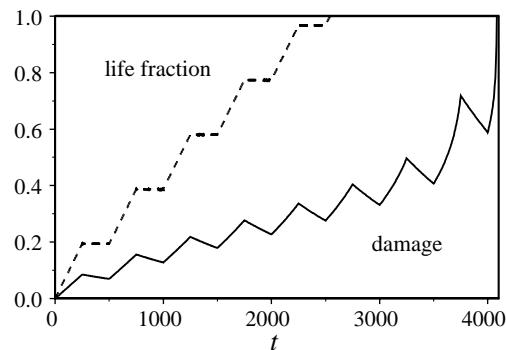


Fig. 5. Accumulation of creep damage (solid line) and life-fraction (dashed line) during a creep-fatigue test using the following data: $E = 1000$, $E_c = 0.8$, σ alternating between 18.5 and zero during $\Delta t = 250$ for each stress level. The time to fracture is $t_U = 4081$ while the life-fraction rule gives $t_{Ro} = 2542$.

The dotted lines show the evolution of damage and strain if the stress is set to zero for $t \geq 800$. As visible in the damage graph, ω heal after this point. The damage will eventually heal completely. As expected, the strain equals zero for $t > 800$ since no plasticity is considered.

In a creep-fatigue test, the stress is altered. A simple, engineering method to calculate the life during alternating loading is provided by Robinson's life-fraction law, cf. Robinson (1952),

$$\int_0^{t_{Ro}} \frac{dt}{t_R[\sigma(t)]} = 1, \quad (90)$$

where $t_R(\sigma)$ is the time to fracture under the constant stress σ and t_{Ro} is the time to fracture under alternating stress according to the life-fraction rule. As noted already by Odqvist and Hult (1962), the Kachanov damage concept implies the Robinson life-fraction rule. It has also been noted that the life-fraction law sometimes underestimates the life, t_U , cf. e.g. Jansson (1985).

Figure 5 shows a test where the stress varies between the levels $\sigma = 18.5$ and zero. The periods of positive stress and zero stress are the same, namely $\Delta t = 250$. Material data are the same as used in the creep test. As shown in Fig. 5, fracture is predicted for $t_U = 4081$ which can be compared to the result using the life-fraction law, $t_{Ro} = 2542$. The result shows the new capability, implicit in the present formulation, to model materials that do not follow simple life-fraction relations. Similar results can be achieved with time-independent models to give the number of load-cycles to fracture that do not follow the Palmgren–Miner life-fraction rule.

5. Discussion and conclusions

Starting with the simplest possible model for isotropic, small strain *thermoelasticity*, a framework for the systematic development of models of plasticity and damage is developed. The thermoelastic model employs constant values of the elastic moduli (K and G) and a constant value of the thermal expansion coefficient (α_v). From a mechanical point of view, this is the simplest possible model capable of modelling straining due to variations in stress and temperature. It is shown that the heat capacity for this model can, at most, be a function of the temperature, $c_v^e(T)$.

The thermoelastic model is first extended into a model for *thermoelastoplasticity* by introducing internal variables (\mathbf{p}) describing the state of internal rearrangement due to plastic deformation. Similarly as in earlier work we assume that the response of the model equals that of the thermoelastic model if \mathbf{p} is constant, cf. e.g. Rice (1971). Under these assumptions, the total strain, the (Helmholtz) free energy and the

entropy split into thermoelastic and thermoplastic parts, cf. Eqs. (35a,b) and (38a,b). It should be noted that this kind of split is allowed for the free energy and the entropy with fewer restrictions on the material parameters than the ones used here. Rosakis et al. (2000) show that it suffices that the specific heat and the stress only depend on the elastic strain and the temperature. In the present paper, the plastic strain and the entropy are shown to be independent of all state variables except \mathbf{p} . The plastic part of the free energy is also shown to depend on \mathbf{p} and, at most, to be a linear function of T , cf. Eq. (45). The plastic stress \mathbf{P} is shown to be dependent of \mathbf{p} and to be, at most, a linear function of T , cf. Eq. (41). Two functions (\mathbf{B}^p and \mathbf{b}^p) of \mathbf{p} have to be given to formulate the dependence. It is noted that a model with temperature independent \mathbf{P} does not produce a permanent change in entropy. The dissipation of the thermoelastoplastic model is derived, cf. Eq. (37a). In this equation the plastic dissipation stress Π is identified. A convenient method to secure a positive dissipation is to introduce a function F , convex in Π , from which the direction of evolution of \mathbf{p} is determined, i.e. $\dot{\mathbf{p}} = \lambda \partial F / \partial \Pi$, with $\lambda \geq 0$ for time-independent plasticity or, $\dot{\mathbf{p}} = \partial F / \partial \Pi$, for creep. A simple example of a time-independent model is provided in Section 4. The model includes a temperature dependent plastic stress. The result is a temperature dependent stress at which yielding start, cf. Eqs. (78)–(80).

In high speed deformation of metals, special consideration is given the fraction β of the rate of plastic work, $\rho \dot{w}_p \equiv \boldsymbol{\sigma} \cdot \dot{\boldsymbol{\epsilon}}^p = \boldsymbol{\tau} \cdot \dot{\boldsymbol{\epsilon}}^p + \sigma_m \dot{\epsilon}_v^p$, converted into inelastic heating, cf. e.g. Rosakis et al. (2000). The inelastic part of the rate of heat is identified from Eqs. (29) and (44) as $\rho \dot{q}_p = T \mathbf{b}^p \cdot \dot{\mathbf{p}} - \Pi \cdot \dot{\mathbf{p}}$. With Eqs. (36a,b) the ratio is given by

$$\beta \equiv \frac{\dot{q}_p}{\dot{w}_p} = \frac{\mathbf{B}^p + T_r \mathbf{b}^p}{\boldsymbol{\tau} \cdot \mathbf{H}^e + \sigma_m \mathbf{H}^e} - 1 \quad (91)$$

Thus, β is generally a function of all the state variables. A result which appears to be in good agreement with experimental results.

Isotropic damage is introduced into the model according to a principle of mechanical and thermal equivalence, cf. Section 3.3. Both the stress and the plastic stress are complemented with the effective stress, $\tilde{\boldsymbol{\sigma}}$, and the effective plastic stress, $\tilde{\mathbf{P}}$, through integrity functions only dependent on the state of damage (ω), cf. Eqs. (46) and (48). For an isotropic model, it is shown that only two independent scalar integrity functions can be introduced for the stress. It should be noted that, in this formulation, ω do not need to be a scalar variable. This result may be viewed as a generalization of the results of Cauvin and Testa (1999). Their representation is however limited to scalar damage variables. With this limitation, isotropic damage can be modelled using, at most, two scalar damage variables. The present formulation opens up more freedom in the modelling. For instance, if constitutive properties are to be determined from simulations, ω can be chosen as a set of parameters determining the geometry of voids and microcracks in a representative volume element.

As shown in Eq. (58), the present principle yields a convenient split of the dissipation in two parts; one part associated with the thermoelastoplastic model and one part associated with the extension of this model to damage. Thus, a thermoelastoplastic model does not need to be reformulated before it is extended to CDM.

The present principle yields a strain split, viz. the total strain is divided into elastic and plastic parts. No strain component corresponding to the state of damage is allowed, although the elastic strain is affected by the state of damage through the integrity functions. It should here be recognized that a strain component corresponding to damage, i.e. ϵ^ω , would lead to severe experimental problems in order to distinguish it from the plastic strain component, ϵ^p . The kinematic relation between \mathbf{p} and the plastic strain is shown to be unaffected by the introduction of damage, cf. Eqs. (54a,b). However, the development of damage will normally affect the elastic strain and the accumulation of plastic strain through the damage-dependence of the stress and the plastic stress. As noted by Ju (1989), the added flexibility due to the introduction of damage is implicitly imbedded in this kind of strain split.

The free energy of the damage model is derived and shown to be split in two parts, the elastoplastic free energy and the cohesive free energy, cf. Eqs. (71a–c). Both parts depend on the damage. The first part

consists of the free energy of the undamaged model modified through the integrity functions. This part can be brought to coincide with different *ad hoc* forms of the free energy through different choices of integrity functions. With one scalar damage variable and with $N^G = N^K = N^P = 1 - \omega$ a free energy corresponding to the models by Ju (1989) and Edlund and Klarbring (1993) results. With N^P equal to an identity tensor, a free energy similar to models by Lemaitre (1985) results. The cohesive part of the free energy is not present in other models; this part is the one that facilitates treatment of healing processes as well as deteriorating processes with positive dissipation. By use of the principle, it is shown that the cohesive part of the free energy only depends on the state of damage and the temperature, cf. Eq. (71c). The dependence is generally given by two functions (\mathbf{B}^c and \mathbf{b}^c) of the state of damage. The temperature dependence is, at most, linear.

Similarly as for the thermoelastoplastic model, the entropy of the damaged model is split in two parts, an elastoplastic part that is identified as the entropy of the undamaged model modified by the state of damage, and a cohesive part only dependent on the state of damage, cf. Eqs. (66a,b) and (69).

The formulated principle leads to a split of the damage stress into an elastoplastic damage stress, $\mathbf{\Omega}^{\text{ep}}$, and a cohesive damage stress, $\mathbf{\Omega}^c$. According to the principle, both $\mathbf{\Omega}^{\text{ep}}$ and $\mathbf{\Omega}^c$ are, at most, linear in the temperature. The elastoplastic damage stress corresponds to the damage energy release-rate of the models described in the introduction. The cohesive damage stress, on the other hand, is a novelty of the present formulation. It can be interpreted as corresponding to the cohesive energy absorbed by the microscopic cracks and voids responsible for the damage. In order to increase damage, the release of elastoplastic energy must overcome the absorbed cohesive energy. On the other hand, in order to decrease damage (healing), the release of cohesive energy must overcome the absorbed elastoplastic energy. If the absorption equals the release, damage evolution is reversible and will not produce dissipation nor permanent damage, i.e. the damage will heal upon unloading. This situation resembles the growth of a Griffith-crack where the size of the crack is always in thermodynamic equilibrium with the acting loads, cf. e.g. Lawn (1993). By requiring that $-\mathbf{\Omega} \cdot \dot{\boldsymbol{\omega}} > 0$, dissipation is related to the deteriorating or healing process and the process is irreversible. There are different ways to assure that the dissipation is non-negative for all possible situations. Within the notion of a Generalized Standard Material, e.g. Nguyen (2000), a convex dissipation potential M is defined such that $\dot{\boldsymbol{\omega}} = -\dot{\mu} \partial M / \partial \mathbf{\Omega}$ with $\dot{\mu} \geq 0$ for time-independent damage or $\dot{\boldsymbol{\omega}} = -\partial M / \partial \mathbf{\Omega}$ for creep damage.

It is argued in Section 3.3 that the heat capacity depends on damage in a specific way, cf. Eq. (63). It is interesting to note that if we had chosen a damage independent heat capacity, or used an integrity function to define an “effective” heat capacity, a non-intuitive elastoplastic damage stress would result. The second term in Eq. (72), here giving no contribution if $\epsilon_v^T = \epsilon_v^c$ or equivalently if $\sigma_m = 0$ (cf. Eq. (27b)), would give a contribution to $\mathbf{\Omega}^{\text{ep}}$ proportional to $\epsilon_v^c(\epsilon_v^c - 2\epsilon_v^T)$ with any of the other choices of heat capacity. Thus, $\mathbf{\Omega}^{\text{ep}}$ would be non-zero with $\boldsymbol{\sigma} = \mathbf{p} = \mathbf{0}$.

It should also be stressed that the cohesive part of the entropy, s^c , is, at most, a function of ω . The fact that the cohesive part of the entropy is independent of the temperature is a consequence of using integrity functions which are functions of the damage variables only. It can also be noted that a damage model with temperature independent damage stress, i.e. $\mathbf{b}^c = \mathbf{0}$, must have $s^c \equiv 0$. Growth of damage will, for such a model, not be accompanied with a change of the cohesive part of the entropy. However, with at least one of α_v , \mathbf{b}^p or c_v^c non-zero, the thermoelastoplastic part of the entropy may change, cf. Eq. (66b).

A simple example of a time-dependent damage model is provided in Section 4. The example shows time-dependent healing at unloading. It is shown that this feature provides a possibility to model materials that do not follow the simple life-fraction rules of Palmgren, Robinson and Miner.

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